1965

An investigation of energy addition to a compressible gas via the hydraulic analogy.

Sloan, John H.

Monterey, California: U.S. Naval Postgraduate School

http://hdl.handle.net/10945/12427

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by

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

IN

AERONAUTICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California
1 9 6 5

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This work is accepted as fulfilling
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MASTER OF SCIENCE
IN
AERONAUTICAL ENGINEERING

from the
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ABSTRACT

The hydraulic analogy to the isentropic flow of a compressible fluid has long been recognized. This study was undertaken to find some means, if possible, of extending the analogy so as to simulate the addition of heat in compressible fluid flow. It was shown that while the analogy does not hold for this phenomenon or for any non-isentropic process, a method does exist by which the isentropic addition of energy may be simulated.

A secondary aim of the investigation was to study and improve the performance of the laboratory water table of the Department of Aeronautics, United States Naval Postgraduate School. The objective was to make the equipment suitable both for flow visualization demonstrations and for further research purposes.

The water table was rebuilt to include a lighted surface incorporating a two-inch grid for photographic reference. An accurate method of measuring local and reservoir water depths was established.

Photographs were taken of flow about models of various shapes to demonstrate the usefulness of the water table and the hydraulic analogy in the study of such flows.

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LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

a	wave propagation speed in an incompressible fluid in a shallow channel
A	area of a duct
c	wave propagation speed in a compressible fluid (sonic velocity)
d	sluice setting
F	Froude number
g	the gravitational constant
h	the enthalpy of a gas
k	the specific heat ratio of a compressible fluid, c_p/c_v
M	Mach number
$\dot{m}$	flow rate
p	pressure
R	the universal gas constant
s	the entropy of a compressible fluid
T	absolute temperature
V	velocity
w	width of an open channel
y	distance in the vertical direction for an incompressible fluid; its depth
z	distance in the vertical direction for a compressible fluid
ρ	density
$(\ )_o$	stagnation conditions (total temperature, enthalpy, etc.)

1. Introduction

The hydraulic analogy has long been recognized as a qualitative and, to a limited extent, quantitative aid in the study of compressible fluid flow. [2] Its basis is the similarity of the equations governing an incompressible fluid flowing in a channel with a free surface and those which describe the flow of a compressible gas. Its usefulness lies in the fact that supersonic flow may be simulated about various shapes for prolonged periods of time at very low cost. It also allows the operator to make numerous variations in the flow patterns quite rapidly and inexpensively.

It was desired to find a method by which the analogy could be extended to include the addition of energy to the stream. Results analogous to those obtained by heating a gas flowing in a constant area duct (Rayleigh line) were expected and, it was thought, a phenomenon analogous to choked flow might develop.

Coincident with the above investigation, the water table of the Department of Aeronautics, United States Naval Postgraduate School, was to be refurbished for use as a teaching aid and as a tool for later investigations of various supersonic flow phenomena.

2. The Hydraulic Analogy

The hydraulic analogy to two-dimensional frictionless isentropic fluid flow has its basis in the similarity of the equations of motion governing the compressible flow situation and those governing the quasi-two-dimensional flow of an incompressible fluid in a horizontal channel with a free surface.

The latter is called quasi-two-dimensional since there are depth changes from point to point, but vertical fluid velocities and accelerations are small and are neglected. [1]

The energy equations governing the two flows are examined below.

Compressible Fluid	Incompressible Fluid
$h + V^2/2 + gz = h_o = \text{const.}$	$p/\rho g + V^2/2g + y = y_o = \text{const.}$
Assuming horizontal flow, z = a constant. Solving for V,	Assuming p constant at all points and solving for V,
$V = \sqrt{2(h_o - h)}$	$V = \sqrt{2g(y_o - y)}$

The continuity equations for the two flows are also analogous. This has been shown to be true for two dimensional flow [4] , but for simplicity it is shown below for the one-dimensional case.

Compressible Fluid	Incompressible Fluid
$\rho AV = \dot{m} = \text{constant}$	$\rho Vwy = \dot{m} = \text{constant}$

Letting $h = c_pT$, and re-writing the energy equation for a compressible fluid in terms of T, k, and M, it is seen that

$$T_o/T = 1 + \frac{k-1}{2} (M)^2$$

The result of re-writing the energy equation for the compressible fluid is, in terms of y and F,

$$y_o/y = 1 + \tfrac{1}{2}(F)^2$$

It may now be seen that the analogy holds if the compressible fluid is such that k = 2.0. This value of k is peculiar to a channel of rectangular cross section. Alternative cross

sections allow other values of k. A triangular cross section, for instance, corresponds to $k = 1.5$. Use of these alternative cross sections has been discussed by Shapiro [7] and [9].

Pressure ratio, p/p_o, is analogued by $(y/y_o)^2$. This is shown by an examination of the equation of state, $p = \rho RT$. Since y is the analogue of both ρ and T, y^2 must be analogous to p.

The expression for wave propagation speed (sonic velocity) in a compressible fluid is:

$$c = \sqrt{KRT}$$

The wave propagation speed for the incompressible situation is:

$$a = \sqrt{gy}$$

This expression is not valid in the ocean or in deep channels, since the assumption of large wave lengths compared to depth is explicit in its deviation. [4]

Thus dimensionless velocity rations may be written for both cases.

Compressible Fluid	Incompressible Fluid
$M = V/c$	$F = V/a$

It is of interest to note that abrupt transition from flow having a Froude number greater than one (supercritical or shooting flow) to a flow having a Froude number less than one (subcritical, tranquil, or streaming flow) is accompanied by a hydraulic jump. This is somewhat analogous to the shock wave which accompanies sudden transition from supersonic to

subsonic flow. It is not completely analogous, since the stagnation temperature of a compressible gas remains constant through the shock, while the total head y_o decreases across a jump due to the dissipation of mechanical energy in the turbulence of the jump. Figure 1 compares the conditions prevailing on both sides of a shock wave with those on both sides of the hydraulic jump.

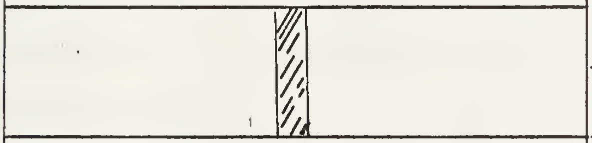
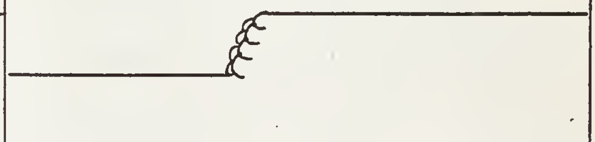
Normal Shock		Hydraulic Jump	
			
h_1	h_2	V_1	V_2
V_1	V_2	y_1	y_2
ρ_1	ρ_2		
p_1	p_2		
s_1	s_2		

FIGURE 1

ANALOGY BETWEEN NORMAL SHOCK AND HYDRAULIC JUMP

Although the momentum equation has not been used in the establishment of the analogy, it may be employed along with the equations of energy and continuity to determine the decrease in y_o across the jump. Assuming the hydrostatic pressure to vary linearly with depth the average pressure on a face perpendicular to the direction of flow will be $\rho g y/2$. The momentum equation can now be written (assuming negligible external friction):

$$\frac{\rho g y_{wy}}{2} + \rho wyV^2 = \text{constant}$$

Expressing velocity in terms of the Froude number, $V^2 = a^2F^2$, where $a^2 = gy$. Making these substitutions and combining the constant terms ρ, g, and w, the equation can be re-written:

$$y^2(1 + 2F^2) = \text{constant}$$

The continuity equation can also be expressed in terms of y and F:

$$y^{\frac{3}{2}}F = \text{constant}$$

Raising the continuity equation to the second power and dividing by the momentum equation to the $\frac{3}{2}$ power yeilds the expression:

$$\frac{F^2}{(1+2F^2)^{3/2}} = \text{constant}$$

This expression is double valued except when F = 1.0 The equation must hold across the hydraulic jump, since the only restriction imposed upon it is that wall friction be negligible. The equation can best be solved graphically by plotting the function $F^2/(1+F^2)^{\frac{3}{2}}$ against F. Two values thus obtained, for example, are F = 3.0 and 0.41. See figure 2.

The energy equation may now be used to find y_{o_2}. Since total energy is not constant across the jump, the energy equation must be written twice.

$$y_{o_1} = y_1 + \frac{V_1^2}{2g} \; ; \quad y_{o_2} = y_2 + \frac{V_2^2}{2g}$$

Substituting $V^2 = gyF^2$ and dividing, it is seen that

$$\frac{y_{o_2}}{y_{o_1}} = \frac{y_2(1+F_2^2/2)^{\frac{1}{2}}}{(1+F_1^2/2)^{\frac{1}{2}}}$$

Since, from the continuity equation, $y_2/y_1 = (F_1/F_2)^{2/3}$, the

ratio of the energy equations can be written:

$$\frac{y_{o_2}}{y_{o_1}} = \frac{F_1}{F_2}^{2/3} \sqrt{\frac{1+F_2^2/2}{1+F_1^2/2}}$$

Substituting the values of F_1 and F_2 previously determined (3.0 and 0.41) y_{o_2} was calculated to be 0.74 y_{o_1}, a decrease of 26%.

It is seen that the decrease in y_o across an hydraulic jump is considerable at high Froude numbers, and of interest to note that a method of establishing this decrease was not found in the literature. The momentum equation was not applied nor even mentioned, although it is satisfied and is analogous to the momentum equation of compressible fluid flow.

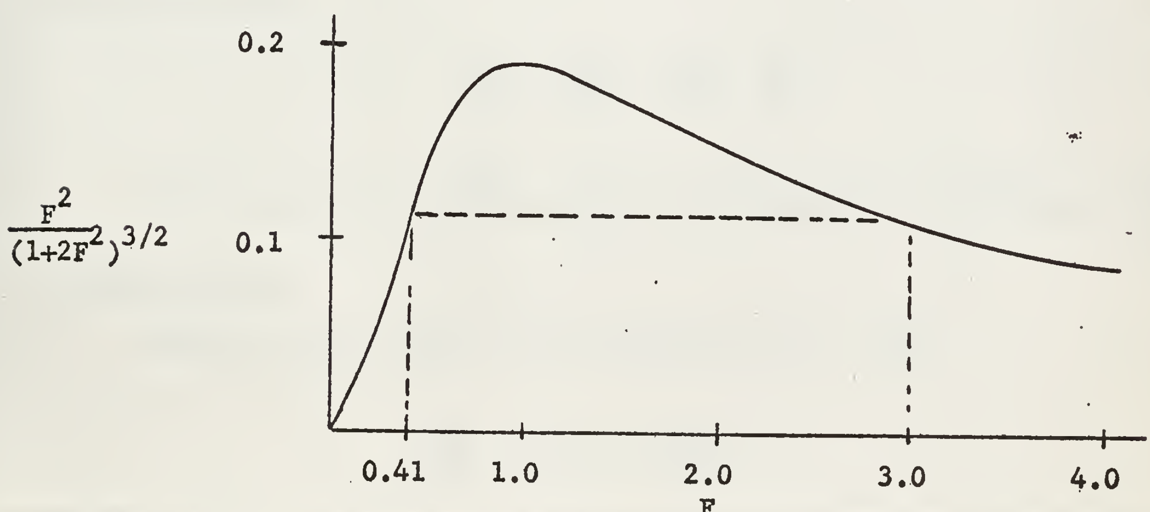


FIGURE 2

COMBINATION OF MOMENTUM AND CONTINUITY EQUATIONS ACROSS AN HYDRAULIC JUMP

The literature refers to the "hydraulic analogy to <u>isentropic</u> compressible fluid flow", but does not offer a proof that this restriction must be imposed.

An examination of the laws of thermodynamics proves that the analogy <u>is</u> restricted to isentropic processes. The combined first and second laws can be written

$$Tds = dh - vdp$$

Utilizing the relationship $dh = c_p dT$, the combined first and second laws can be re-written:

$$ds = c_p \frac{dT}{T} - vdp/T$$

Substituting $v/T = R/p$ from the equation of state and dividing through by R

$$\frac{ds}{R} = \frac{c_p}{R}\frac{dT}{T} - \frac{dp}{p}$$

Since $R = c_p - cv$, and $c_{p/cv} = k$, $c_p/R = \frac{k}{k-1}$. Making these substitutions,

$$\frac{ds}{R} = \frac{k}{k-1}\ \frac{dT}{T} - \frac{dp}{p}$$

The quantities $\frac{dT}{T}$ and $\frac{dp}{p}$ are analogued by $\frac{dy}{y}$ and $2y\ \frac{dy}{y^2}$ respectively.

Substituting into the expression for $\frac{ds}{R}$,

$$\frac{ds}{R} = 2\frac{dy}{y} - 2\frac{dy}{y} \equiv 0.$$

Thus it is seen that the analogy holds only for isentropic processes.

Table I summarizes the analogous quantities.

TABLE I

ANALOGOUS QUANTITIES BETWEEN A COMPRESSIBLE GAS AND AN INCOMPRESSIBLE FLUID

Compressible gas	Incompressible Fluid
T/T_o	y/y_o
ρ/ρ_o	y/y_o
p/p_o	$(y/y_o)^2$
$V = \sqrt{2c_p(T_o - T)}$	$V = \sqrt{2g(y_o - y)}$
$c = \sqrt{kRT}$	$a = \sqrt{gy}$
$\dot{m} = \rho AV$	$\dot{m} = \rho wyV$
$M = v/c$	$F = v/a$
shock wave	hydraulic jump

3. Addition of Energy

Energy is usually added to a compressible gas flowing in a duct by burning fuel in the stream, thus increasing the total temperature. This process is not isentropic and cannot be simulated via the hydraulic analogy. The results of an attempt to "force" an analogy are discribed below.

Energy can be added to the water at any point in the stream by dropping the level of the surface upon which it is flowing. This causes an effective increase in y_o with a corresponding change in velocity and Froude number.

The energy and continuity equations can be combined to give an equation relating local depth y and stagnation depth

y_o at any fixed flow rate and channel width.

The energy equation is written

$$V^2 = 2g(y_o - y)$$

and the continuity equation

$$\rho w y V = \dot{m}$$

or

$$y^2 V^2 = \left(\frac{\dot{m}}{\rho w}\right)^2$$

Substituting for V^2 from the energy equation

$$y^2[2g(y_o - y)] = \left(\frac{\dot{m}}{\rho w}\right)^2$$

Let

$$\xi = \sqrt[3]{\frac{1}{2g}\left(\frac{\dot{m}}{\rho w}\right)^2}$$

then

$$y^2 y_o - y^3 = \xi^3$$

This relationship is shown in Figure 3.

The two branches of the curve shown in Figure 3 represent flow at subcritical and supercritical Froude numbers. Flow occuring at values of y and y_o corresponding to the "nose" of the curve is at a Froude number of 1.0. Under these conditions $y = 2/3\ y_o$.

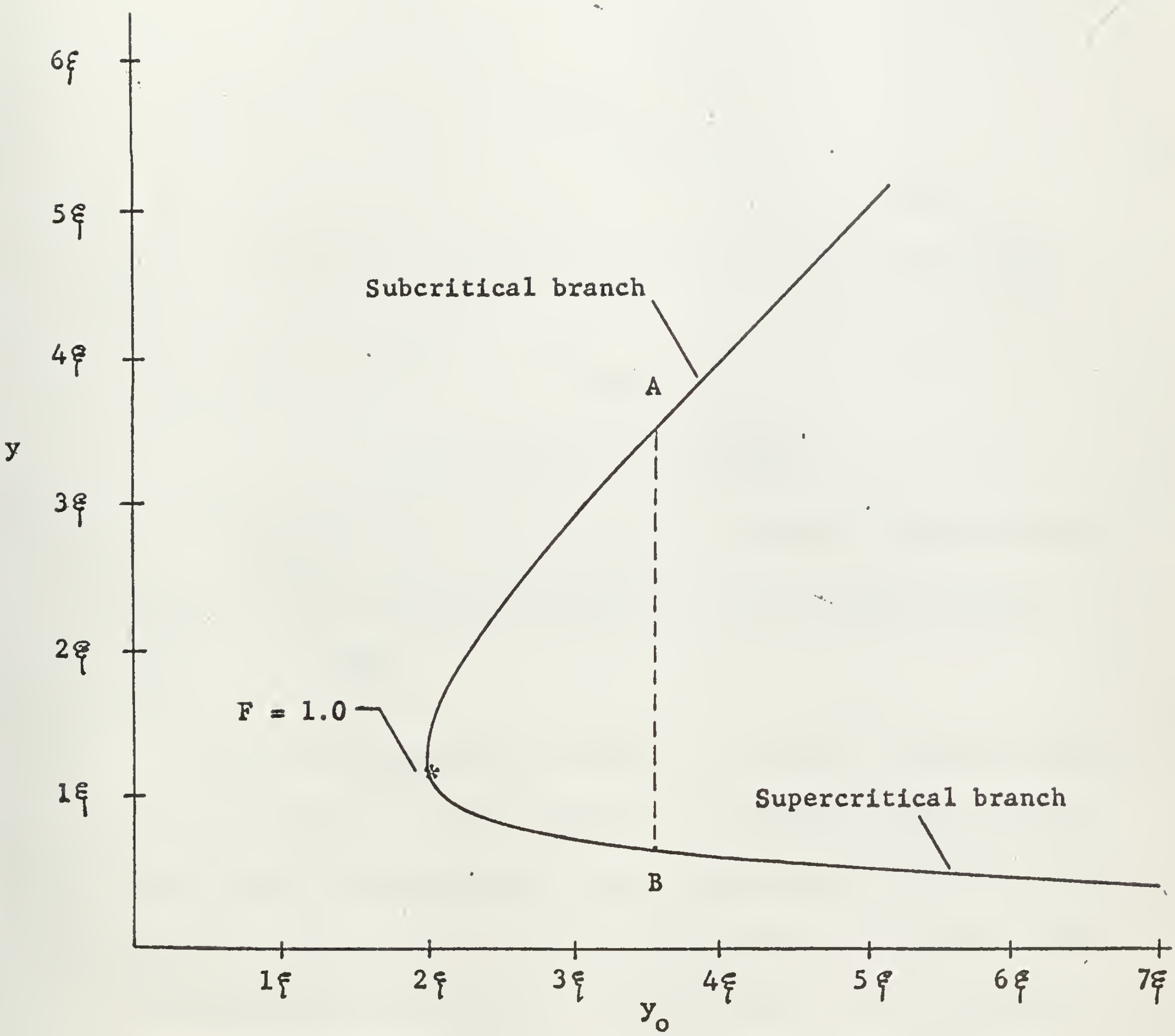


FIGURE 3

COMBINATION OF ENERGY AND CONTINUITY
EQUATIONS FOR AN INCOMPRESSIBLE FLUID

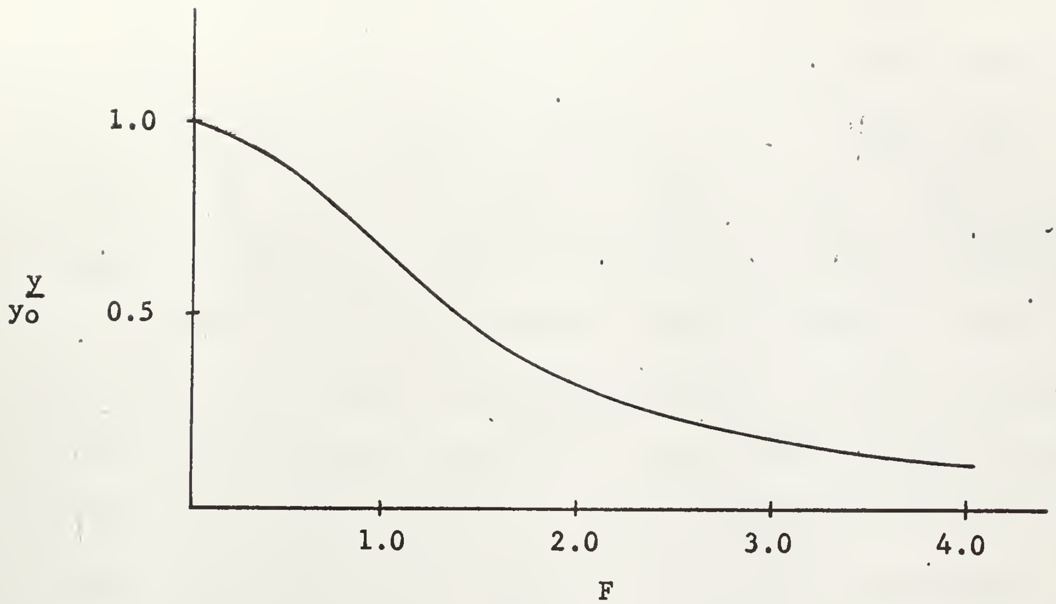


FIGURE 4

y/y_o VERSUS FROUDE NUMBER

Figure 4 is a plot of y/y_o against Froude number, based on the non-dimensialized energy relationship,

$$y/y_o = \frac{1}{1+F^2/2} .$$

The supercritical branch of Figure 3 shows there is no analogy to the "choked" condition which occurs when heat is added to a compressible gas flowing in a constant area duct. As y_o increases, y decreases and Froude number increases without limit. In the non-isentropic addition of heat to a compressible gas, T increases as T_o is increased and Mach number approaches 1.0 as a limit. Choking occurs when the product ρV is a maximum, at which point M 1.0. Since the analogue of

ρV is Vy which remains constant, there will be no analogue to choked flow.

The subcritical branch indicates that as y_o is increased y also increases. This can be visualized by thinking of water in a shallow stream running over a slight drop with little or no apparent change in water level. Here y_o is increased and so is y. Velocity decreases as does Froude number.

If the drop is made great enough, however, there will be a decrease in y, and the velocity after the drop becomes greater than the velocity before it. The Froude number also increases from subcritical to supercritical. This must correspond to a shift from point A to point B as shown in Figure 3 since flow at these points can occur on either branch at the same y_o. The magnitude of the increase in y_o required to produce this shift is thought to be a function of downstream conditions such as channel friction and is a subject for further investigation. It was not carried out at this time since to do so would have destroyed the table surface.

Another method of investigating the effect of an increase in y_o on the subcritical branch was attempted. It consisted of forcing an increase in y_o by lowering the sluice and measuring the resultant increase in y.

No increase in y was noted as the sluice was lowered, even though the initial sluice position was clear of the flow. This was attempted at flow rates varying from 15 to 30 gallons per minute. These flow rates caused Froude numbers from 0.4 to 0.65 with the table level and the sluice in the clear position.

It is felt that the effect expected was not observed due to several factors, the foremost being the limited capacity of

the pump. Operating at a flow rate of 30 gallons per minute with the sluice lifted clear of the flow, a velocity of 0.68 feet per second was achieved with the table level. This corresponds to a reservoir depth (y_o) of 0.488 inches and a stream depth (y) of 0.406 inches. Even though the difference of 0.088 inches is measurable with the apparatus described in Section 4, any increase in y caused by contact of the sluice with the water surface was not. Any further depression of the sluice caused a shift to supercritical flow immediately downstream of the sluice. A hydraulic jump then occured immediately and the flow was back to subcritical.

It is felt that if larger flow rates were possible the increase in the quantity (y_o-y) for subcritical flows would allow the demonstration of the increase in y with y_o described by the equations of motion and continuity.

This increase is shown diagramatically in Figure 5 as it is expected to appear. The flow in Figure 5 (a) is subcritical and the sluice is set at d_1. As the sluice is depressed, in Figure 5 (b), d decreases to d_2 and y increases to y_2. The Froude number also decreases. As the sluice is depressed further to some setting d_3, transition to supercritical flow occurs at y_{o3} and y_4 with a Froude number greater than 1.0 as shown in Figures 5 (c) and 5 (d).

Another method of increasing the energy of the stream would be to add water at some point in the stream. If this water were at a higher head than the original y_o the resultant flow would have a higher head per unit mass than the original stream.

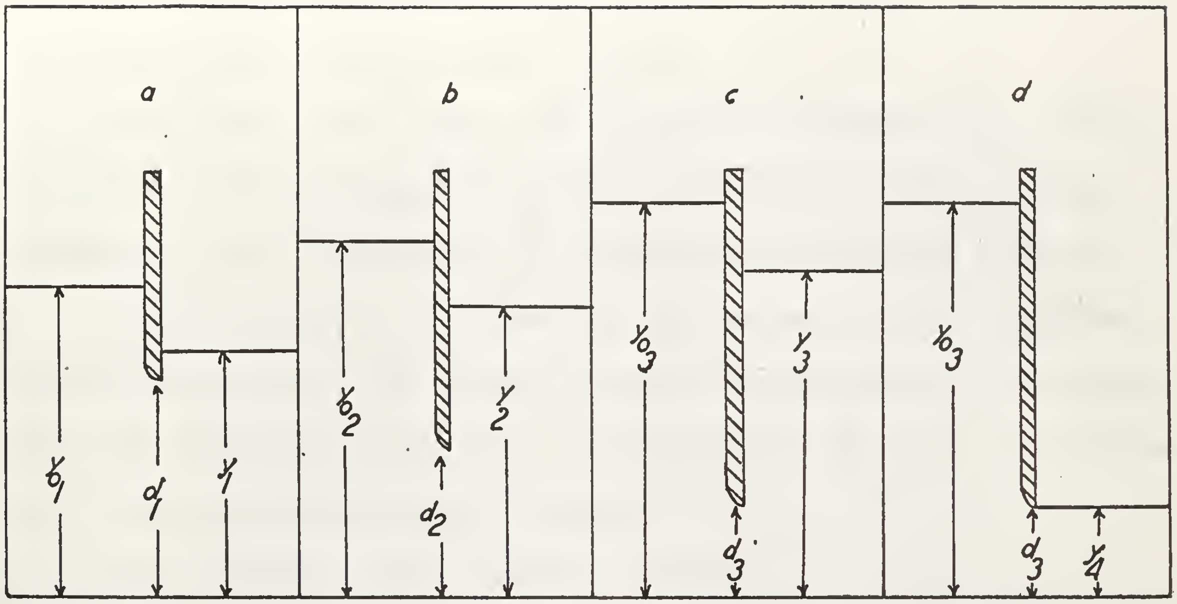


FIGURE 5

FLOW AT VARIOUS SLUICE SETTINGS

To determine if this process is analogous to the heating of a compressible fluid flowing in a duct, consider the case where the duct area is varied as heat is added. Let the area vary such that velocity remains constant. From the continuity equation, it is seen that density must decrease if the product ρAV is to remain constant. If velocity is constant, the energy equation states that the quantity $(T_o - T)$ must be constant. Since heat is being added and T_o is increasing, T must be increasing at the same rate.

Now consider the incompressible situation. If the analogue is to hold, the parameters analogous T and ρ

must increase and decrease, respectively. This is impossible since both are analogued by y.

4. Description of the Water Table.

The water table used was initially constructed by Naval Ordnance Test Station, China Lake, California The design was based upon the experience of the General Electric Company. [5]

Constructed of wood and painted with an acid resistant laboratory paint, the table measures approximately 42 inches wide by 69 inches long. The legs are fitted with jack screws to facilitate table slope changes.

The original table bed was replaced by a sheet of lucite one inch thick, 36 inches wide and 48 inches long. The underside of the lucite was frosted and a two inch grid drawn upon it. Three four foot flourscent light tubes were mounted under the table in a direction parallel to the flow. The resulting illuminated grid is suitable for photography. The sides of the channel are made of five-eighth inch lucite and are bonded to the table to insure a water tight fit. An adjustable sluice was constructed in order to regulate y_0 at any given flow rate. The bottom edge of the sluice was milled to a one-half inch radius to prevent tripping to turbulent flow.

Water flows under the sluice and across the table from a reservoir of three cubic feet capacity. The reservoir is fed by a one inch pipe perforated at random intervals to insure an even distribution of incoming water. Two screens were fabricated from honeycomb stock (one-eighth inch cell). These were inserted in the reservoir and provide complete "stillness"

which insures adequate stagnation conditions. Water enters the channel over a lip which has an eight inch radius to promote gentle transition from zero velocity.

The water is collected in a return trough and fed back to the reservoir through a centrifugal pump and a flowrotor. The maximum flow rate of the system is 30 gallons per minute.

Depth measurements are made by means of a three inch depth micrometer with a pointed tip attached. The micrometer is bolted to an aluminum bar which extends across the table. The bar rests on two steel straps mounted parallel to the channel bed. Measurements made with this apparatus are accurate to plus or minus 0.005 inches. This accuracy is limited by the bow inherent in the lucity bed and by the alignment of the steel straps. Figure 6 shows the table and measuring apparatus.

Detergent was added to the water to reduce surface tension and capillary waves. It was discovered that if enough detergent were added the action of the pump caused a suspension of bubbles. The resulting flow had a gray color and provided excellent contrast for photographs.

An intravenous feeding apparatus was obtained and used to admit drops of dye to the system. Red food coloring (undiluted) proved to be a satisfactory dye. This arrangement could be closely regulated, permitting drops to fall on the flow at one-half second intervals. Another method of measuring drop interval was to time a number of drops, say twenty,

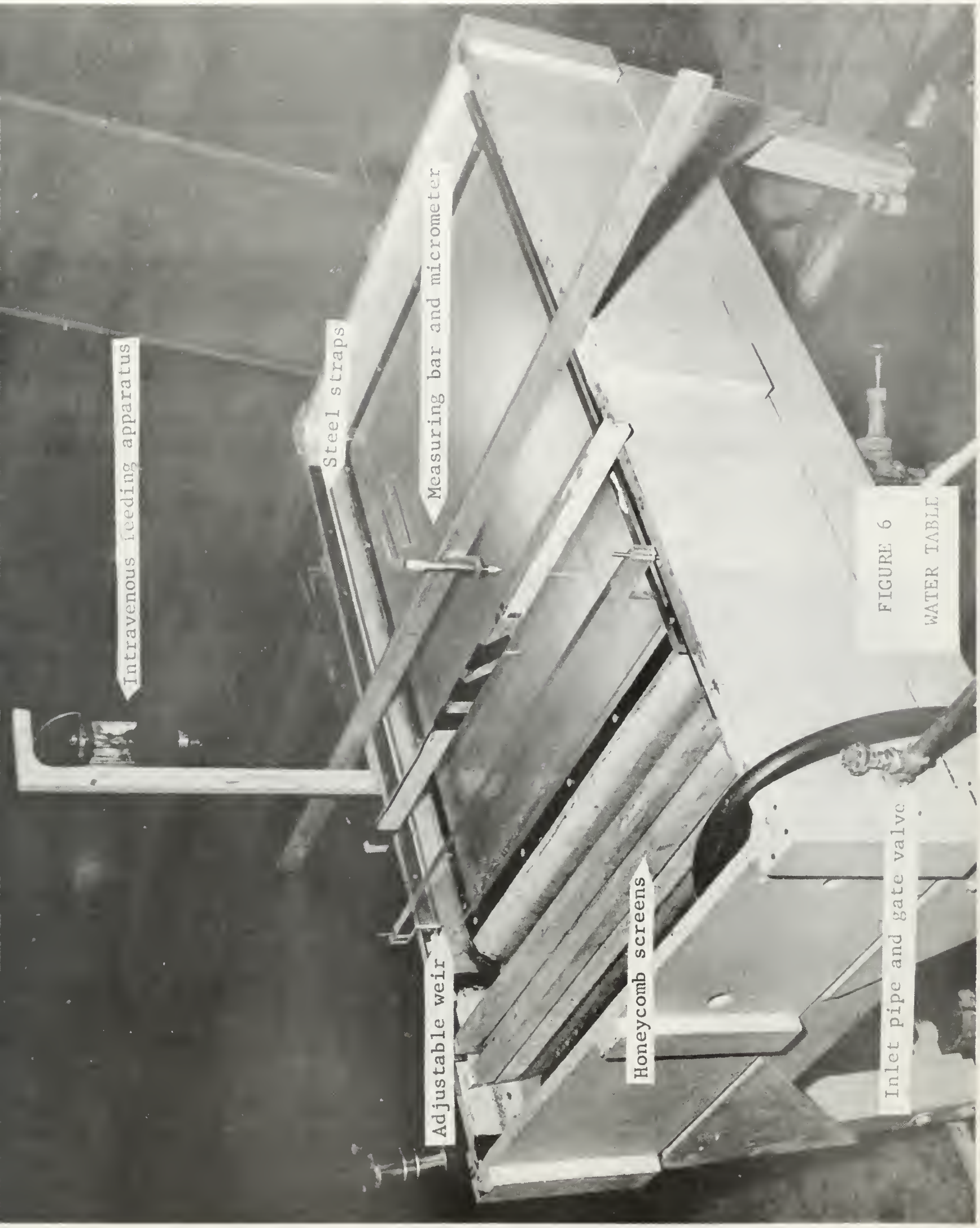


FIGURE 6
WATER TABLE

with a stop watch. The drops were then photographed downstream (against the grid) giving an accurate measurement of velocity. The fact that the head of the dye reservoir is constantly decreasing is insignificant in light of the small amount of dye added at one time. This is limited by the ultimate coloration of the water.

5. Examples of Capabilities of the Table

Photographs were made of flow about various models at several Froude numbers to check the effectiveness of the lighting, dye addition system, and constrast.

Figure 7 shows drops of dye downstream. Twenty drops of dye were added over a period of 12 seconds.

FIGURE 7

VELOCITY MEASUREMENT BY DYE DROP METHOD

The photograph shows the drop 13.2 inches (6.6 grid squares) apart. Thus the velocity of the stream is 1.8 feet per second. Depth measurements taken at the same time indicated the critical velocity was 0.8 feet per second. The Froude number is 2.2.

FIGURE 8

TRAILING VORTICES

Figure 8 is a photograph of the vortices produced by a symmetrical airfoil at a slight angle of attack. The Froude number was 0.85. Soap was allowed to circulate through the pump to produce visible vortices.

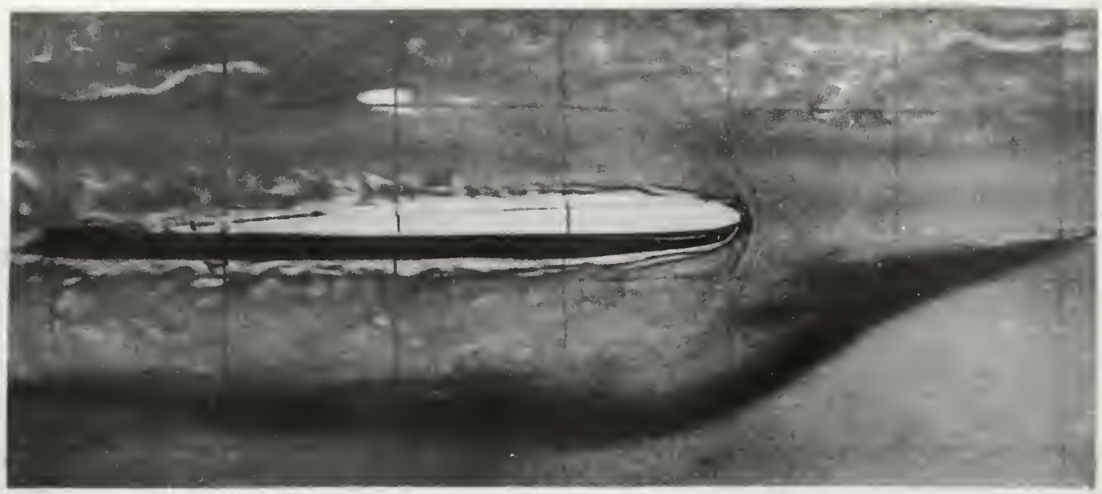


FIGURE 9

STREAMLINE PATTERN PAST A SYMMETRICAL AIRFOIL

The streamline pattern about the same airfoil is shown in Figure 9. Red food coloring was used as a dye.

The flow past a supersonic airfoil is compared to that past a lowdrag subsonic airfoil (of approximately equal thickness ratio) in Figure 10. Note the detached "shock" on the blunt airfoil compared to the attached shock on the wedge shaped airfoil. The Froude number was 2.3.

Figure 10 shows that oblique shock formed by the centerbody of a ramjet intake at F = 3.1. Note that design conditions are being met; the shock is on the

lip of the inlet. Secondary shocks are seen in the diffuser.

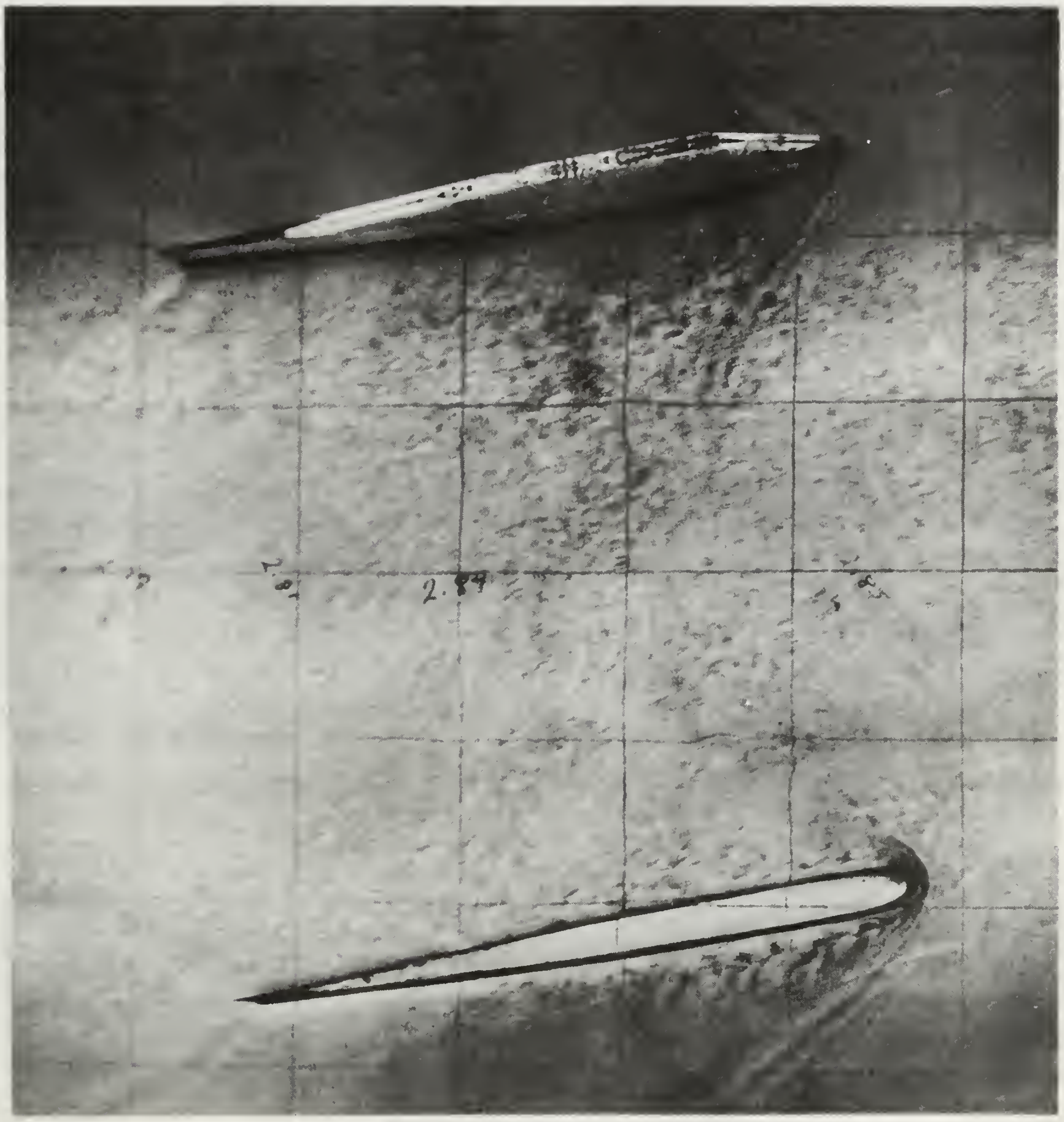


FIGURE 10

"SUPERSONIC" FLOW PAST TWO DIFFERENT AIRFOILS

The hydraulic analogy has also been used to study

FIGURE 11

FLOW THROUGH A RAMJET

cylinder and a converging-diverging nozzle [9], as well as the flow about various other aerodynamic shapes [10]. References [7] and [10] are considered extremely useful in demonstrating the applicability of the analogy and the techniques required in its use.

6. Recommendations

On the basis of knowledge gained during this investigating, it is recommended that a new water table be constructed for use at the United States Naval Postgraduate School. The new table should incorporate a glass bed since the lucite in the present table is easily scratched and not easily polished. It should also be much larger, with the dimensions of the table described in reference [1] considered minimum. Other features considered mandatory are a variable porosity tail gate and a larger settling chamber. The table should also provide for application of the "hydrogen bubble" technique [6]. This would eliminate the necessity for a dye addition system, as the bubbles will give accurate velocity and streamline information.

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